Examples Programming in R

From C programming to R style

Goal

- two football teams with different averages
- how often does the better team win ?
  - poisson processes

Ingredients / Plan

Some ingredients / the plan

- poisson processes / exponential distribution / time between events
- how to code it
  - naïve – time-based simulation
  - changing distributions
  - C programmer version
  - using R features

Taylor expansion of ln x

Will need (soon)

$$\lim_{n\to\infty}\left(1+\frac{T}{N}\right)^N$$

First I want to know about $\ln(x+1)$

Remember Taylor expansion for some a

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

for logarithms

$$\ln(x)=\ln a+\frac{x-a}{a}-\frac{(x-a)^2}{2a^2}+\frac{(x-a)^3}{3a^3}+\cdots$$

why ? do not forget $\frac{d}{dx}\ln x=\frac{1}{x}$

from previous slide

$\ln(x) = \ln a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} + \cdots$

so

$$\ln(x+1) = \ln(a) + \frac{x+1-a}{a} - \frac{(x+1-a)^2}{2a^2} + \frac{(x+1-a)^3}{3a^2} - \cdots$$

let me set $a = 1$

$$\ln(x+1) = \ln(1) + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

what happens as $x \to 0$?

$$\lim_{x \to 0}(\ln(x+1)) = x$$

will need this later

Uniformly distributed events

Decay of a particle / chemistry

- $\mathrm{A} \rightarrow \mathrm{B} + \mathrm{C}$ long term average is clear $A(t) = A_0 e^{-\lambda t}$
- intuitive
  - in some time ΔT I can talk about the probability of a breakdown
  - if the decay rate λ is high, the probability is higher
  - say time between breakdown is $\tau = \lambda^{-1}$
- we rarely look at individual molecules ($\Delta T \gg \tau$)
- when do we see individual events ?
  - football game (and Geiger counters, ion channels)

Non-Uniformly distributed events

- Football – long term average is clear (1300 goals in 1000 games)
- short term ? very uncertain – no goals, 5 goals are possible
- order of magnitude..
  - $\tau = \frac{T}{N} = \frac{90}{2} = 45 \text{ min}$ (for about two goals scored)
- other systems in biology / chemistry ?
  - ion channels in nerves open / close spontaneously (rare, but easy to measure)
  - few copies of DNA repressor per cell
    - DNA + protein → (DNA-protein) rare event – hard to see
    - classical chemical kinetics is not helpful

Distribution for these events

- Start from average over long T
- divide into $N \times \Delta T$
- get limit as $N \to \infty$ and $\Delta T \to 0$

my nomenclature

- rate $\lambda = {}^{1}/_{\tau}$ the average time between goals / channel opening / ..

- what is the average number of goals in ΔT ? $P(\Delta T) = \lambda \Delta T$
- and probability of no goal in some ΔT is $\mathrm{P}_0(\Delta T) = (1 - \lambda \Delta T)$

longer time with many ΔT

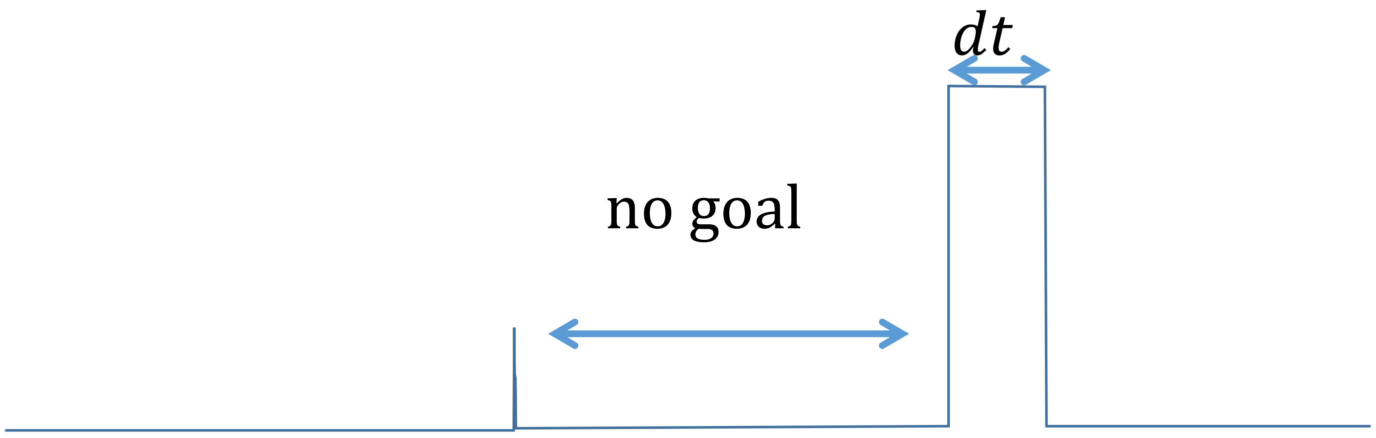
$P_0(T) \approx (1 - \lambda \Delta T)^N$ or $\left(1 - \frac{\lambda T}{N}\right)^N$

Result from earlier …$\lim_{x \to 0} \ln(1 + x) = x$

$$
\begin{aligned}
P_0(T) &\approx \left(1 - \frac{\lambda T}{N}\right)^N \\
&= \lim_{N \to \infty} \left(1 - \frac{\lambda T}{N}\right)^N \\
&= \exp\left(\log \lim_{N \to \infty} \left(1 - \frac{\lambda T}{N}\right)^N\right) = \exp\left(N \log \lim_{N \to \infty} \left(1 - \frac{\lambda T}{N}\right)\right) \\
&= \exp\left(N \frac{-\lambda T}{N}\right) \\
&= e^{-\lambda T}
\end{aligned}
$$

The exponential distribution

- probability for no goal $P_0(T) = e^{-\lambda T}$
- check intuition
- what I want is the probability of 1 goal, 2 goals, … within time t



$I_1 dt =$ probability of no goal in t $\times$ probability of goal in dt

$$I_1 dt = P_0(t)\ \lambda\, dt = \lambda e^{-\lambda T} dt$$

$I_1 = \lambda e^{-\lambda T}$ exponential distribution

One possibility – use exponential distribution

- naïve inefficient simulation

we have rates λ_1 and λ_2 , work out total $\lambda_0 = \lambda_1 + \lambda_2$

set up counters n_1 and n_2

set up tmp_1 and tmp_2

while $(t < T\_game)$ {

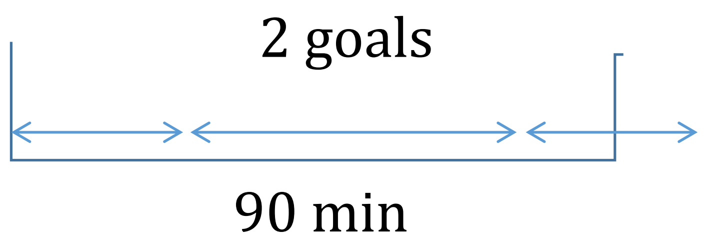
    $n_1 += tmp_1$; $n_2 += tmp_2$;

    $tmp_1 = tmp_2 = 0$

    $\Delta t = random\_from\_exponential(\lambda_0)$

    decide who gets goal (random based on $\frac{\lambda_1}{\lambda_1+\lambda_2}$)

    increment tmp_1 or tmp_2



- we can be much more efficient

expected number of goals in t

- start with binomial distribution
- probability of success in one try is p
- I have n tries
- what is the probability of seeing k successes

$$P(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

- you remember $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- When do you see this ?

Probability of seeing $k = 5$ heads from $n = 10$ coin tosses with $p = {}^1/_2$

- rate per unit time game λ
- some rules
  - events (goals) are independent
  - events are rare – probability of one in short time t is λt
  - you never see two events in a very short time

- take unit time and divide by n (trials)
- probability in one of n units is $p = {}^{\lambda}/_{n}$
- we are interesting in case of very small p in any one δt

original name binomial $P(k|n,p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$

write as $P(X=x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ remember $p = \frac{\lambda}{n}$

consider limit

$$\lim_{n\to\infty} \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{n^x} \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

from binomial to poisson

$$\lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{n^x} \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{n^x} = \lim_{n\to\infty} \left[\frac{n}{n}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right] = 1$$

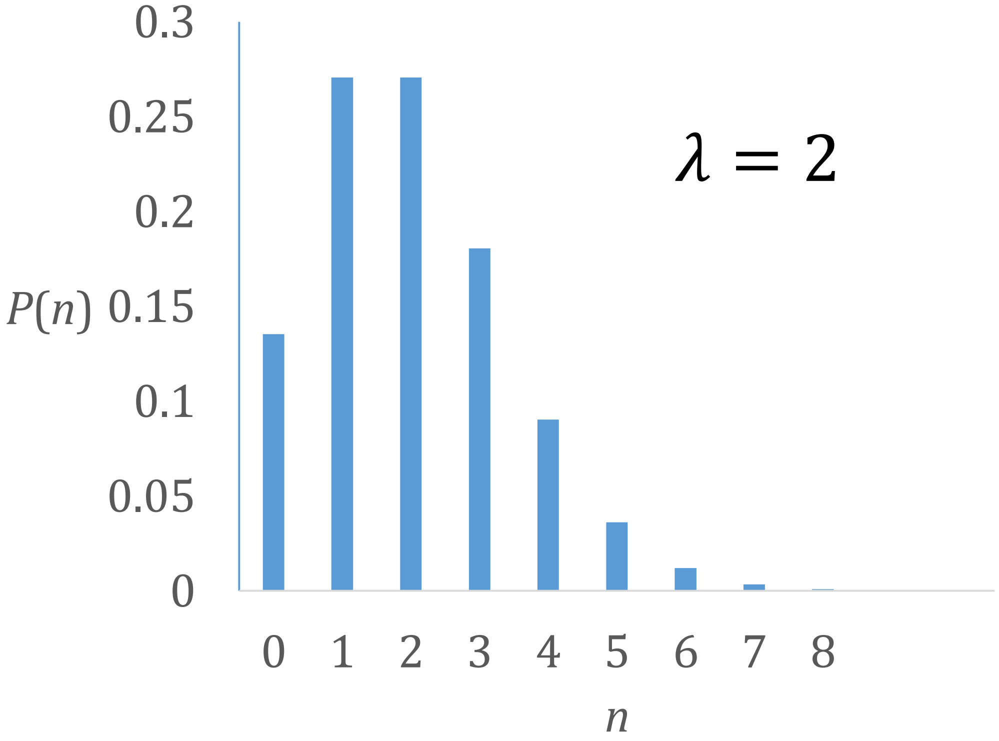
$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \text{ and } \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x} = 1$$

$$\lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x e^{-\lambda}}{x!} = P(X=x)$$

Poisson distribution

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

- from average rate of events λ I can calculate the probability of seeing some number x events
- change simulation strategy



simulation strategy

- look up rate of goals for team 1 (λ_1) and 2 (λ_2)
- say $\text{Pois}(\lambda)$ is a random number drawn from poisson distribution
- a game is

$n_1 = \text{Pois}(\lambda_1)$ and $n_2 = \text{Pois}(\lambda_2)$
if $(n_1 > n_2)$ team 1 wins
elseif $(n_2 > n_1)$ team 2 wins
else draw

- repeat many times to get probabilities

- first approach C style

C programmers version of football

```
game <- function (mu_1, mu_2)  {
    team1_result <- rpois(1, mu_1)
    team2_result <- rpois(1, mu_2)

    if (team1_result > team2_result) {
        result <- 1
    } else if (team2_result > team1_result) {
        result <- 2
    } else {
        result <- 0
    }
    return (result)
}
```

`rpois` random number from Poisson distribution

to run the game..

```
result <- c()
for (i in 1:n_games) {
     result <- c(result, game(team1_mu, team2_mu))
}
w1 = length(result[result==1]); w2 = length (result[result==2])
draw = n_games - (w1 + w2)
cat ("team 1", w1, w1/n_games * 100, "%\n")
cat ("team 2", w2, w2/n_games * 100, "%\n")
cat ("draw  ", draw, draw/n_games * 100, "%\n")
```

fancy indexing
select elements
where results is 2

from 100 000 games

```
team 1 28630 28.6 %
team 2 43422 43.4 %
draw   27948 27.9 %
```

- took 10 ½ s   can do much better

- games are independent events
- use vectors in R

```
team1_mu <- 1.  # Average number goals per match
team2_mu <- 1.3
n_games <- 100000  # How many games to play
```

```
team1 <- rpois(n_games, team1_mu)
team2 <- rpois(n_games, team2_mu)
```

generate 100000 results in one go

team1/2 are long vectors

```
w1 <- sum (team1 > team2)
w2 <- sum (team2 > team1)
draws <- n_games - (w1 + w2)
```

sum over logicals

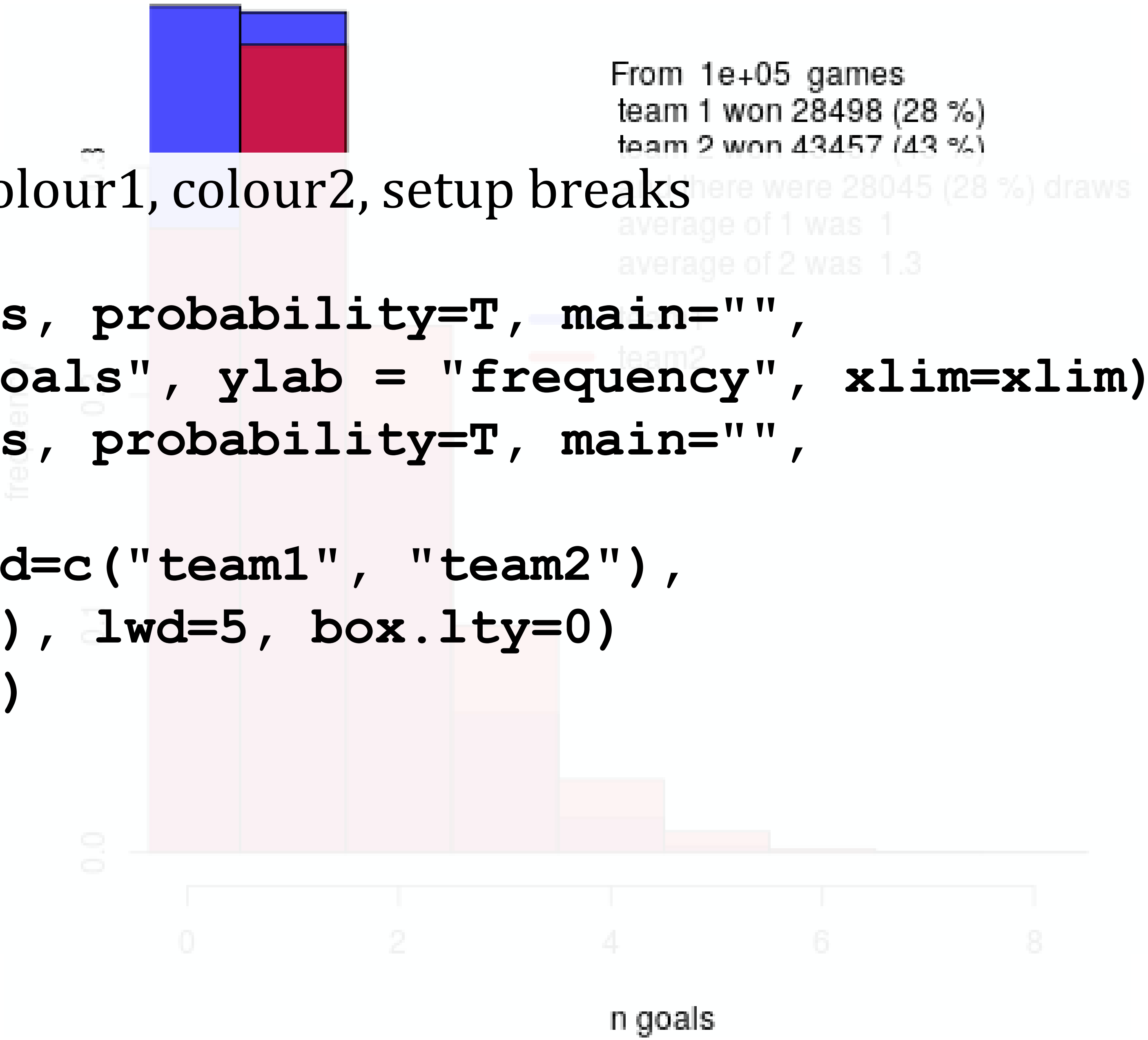
team2 > team1 is a long logical vector

- from 10 ½ s to 0.13 s (including printing results) let us plot results

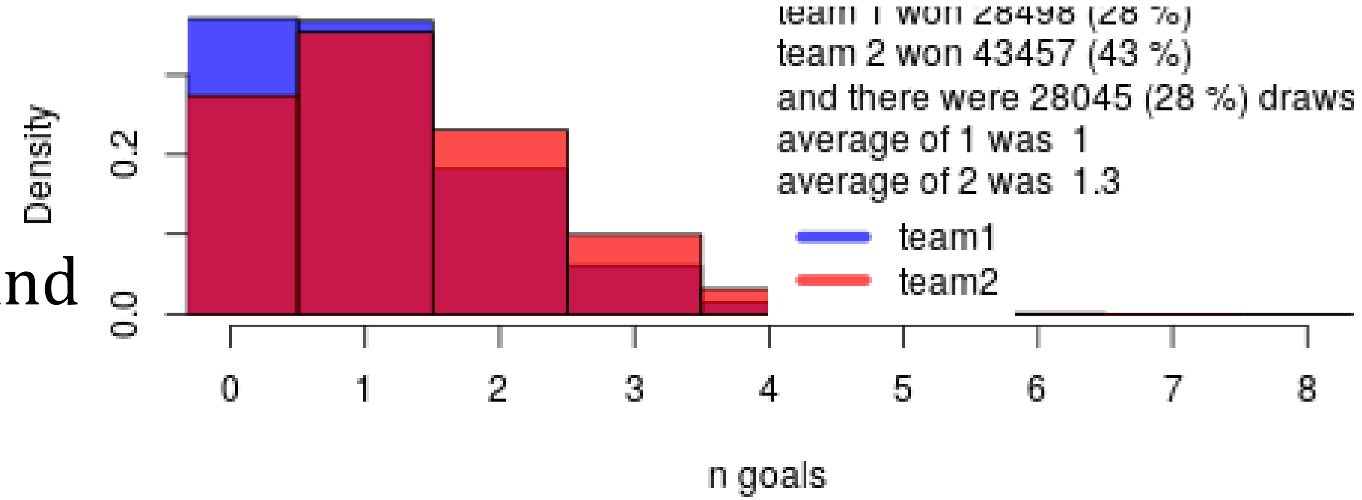
- build up text, put in "a", define colour1, colour2, setup breaks

```
xlim=c(0,9)
hist (team1, breaks=breaks, probability=T, main="",
   col=colour1,xlab= "n goals", ylab = "frequency", xlim=xlim)
hist (team2, breaks=breaks, probability=T, main="",
col=colour2, add=T)
legend(x=3, y=0.25, legend=c("team1", "team2"),
   col=c(colour1, colour2), lwd=5, box.lty=0)
text(a, x=4, y=0.3, adj=0)
```

- can make the plot clearer
  - box and whiskers



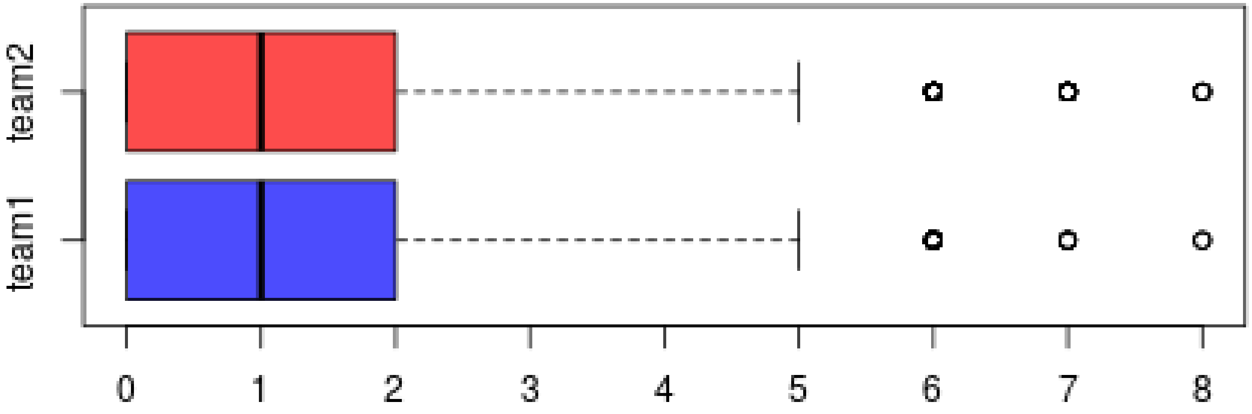
- remember the scores are in team1 and team2
- add a command for two rows and




```
boxplot (df, horizontal=T, ylim=c(0,8), col=c(colour1, colour2))
```

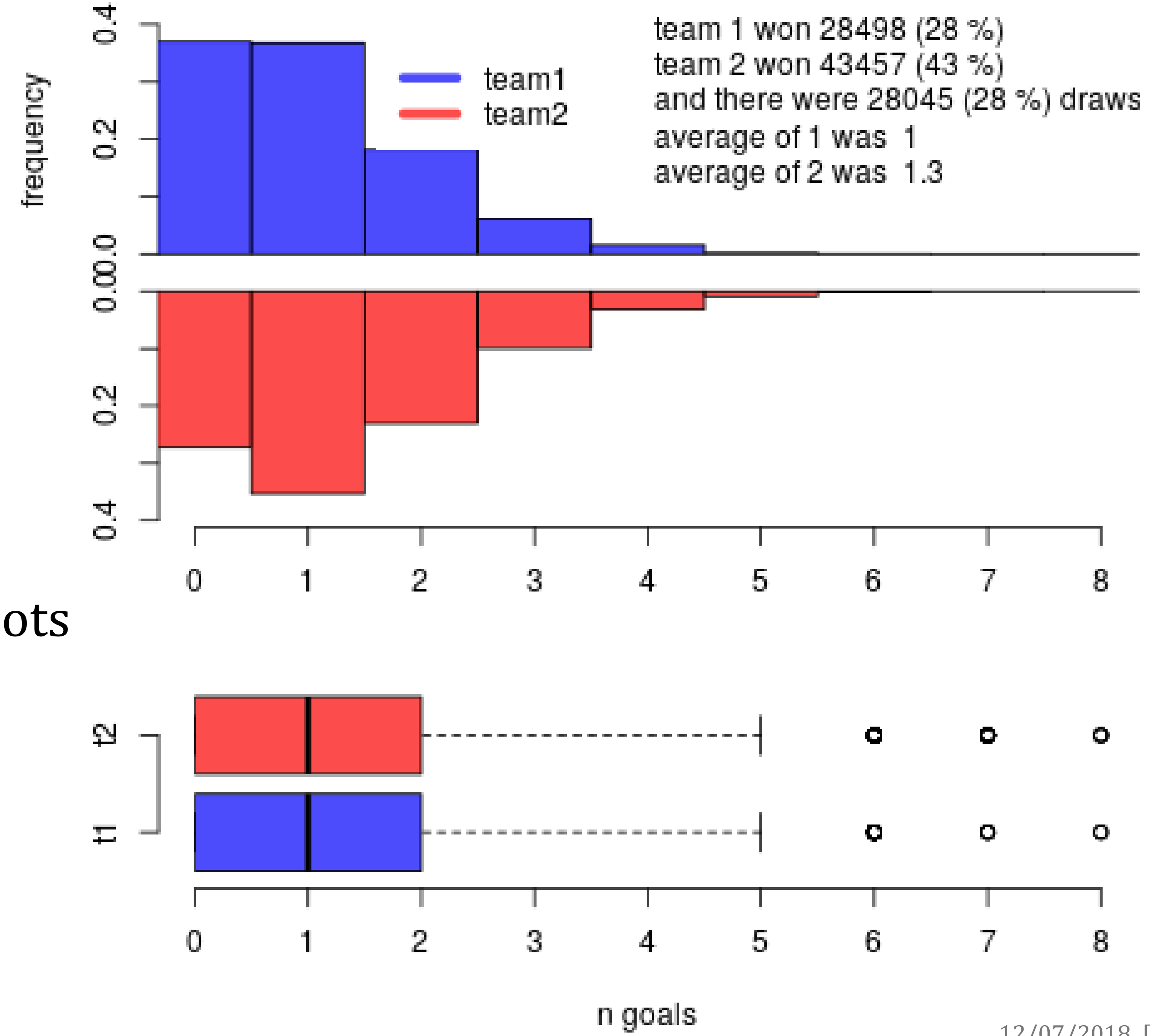
- line shows median
- half box is 25 %
- dots for outliers

- histogram is not so clear



- flip limits on second histogram – no editing of data

- some tedious placement of plots



can we make football better ?

- instead of one time unit $\mu = \lambda$ make games n times longer so
$$\mu = \lambda n$$

- put 100 000 games into a function

```
oneround <- function (mu1, mu2, n_games) {
    team1 <- rpois(n_games, mu1)
    team2 <- rpois(n_games, mu2)
    w1 <- sum (team1 > team2)
    w2 <- sum (team2 > team1)
    draw <- n_games - (w1 + w2)
    return (c(w1, w2, draw))
}
```

- and call this for different values of μ_1, μ_2 scaled by n

more compact

How often does the better team win ?

```
oneround <- function (mu1, mu2, n_games) {
     team1 <- rpois(n_games, mu1)
     team2 <- rpois(n_games, mu2)
     w2 <- sum (team2 > team1)
}
```

just collect results for team 2

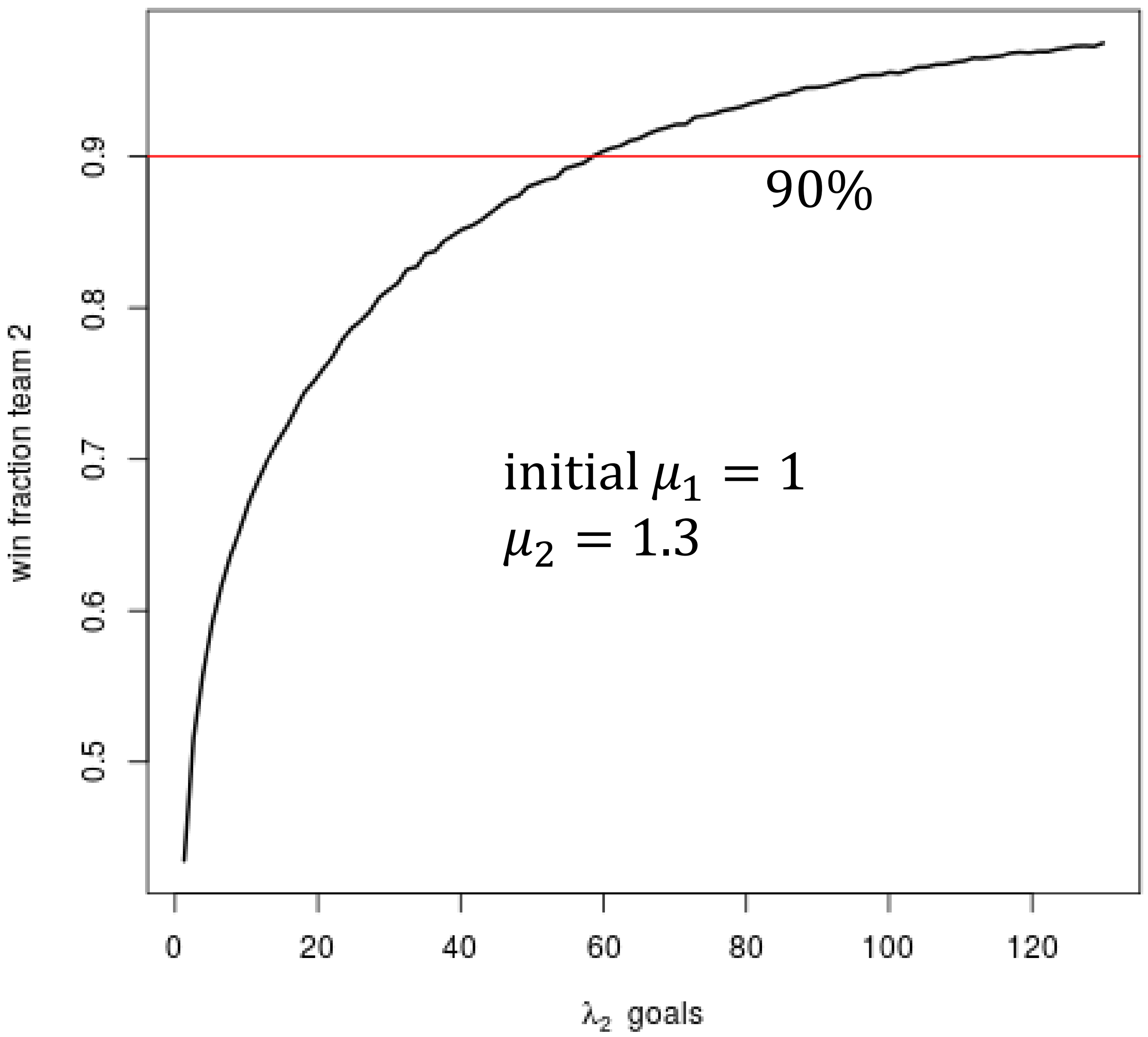
play repeatedly

```
mult <- seq(from = 1, to = 100, by = 1)
wins <- c()
for (m in mult) {
     r <- oneround (team1_mu * m, team2_mu * m, n_games)
     wins = c(wins, r)
}
```

- plot the results

- if game are 10 times longer better team wins 62 %

- if football games are about 60 fold longer, results are interesting



what has one seen ?

Did I cheat in scripts ? not much

- some code for placing plots on page, setting random seed, histogram breaks

Football results are close to meaningless

R programming

- ugly but very powerful – basic poisson competition less than 10 lines
- graphics easy, but syntax horrible
- use built-in functions
- work with vectors not scalars

more serious R

Real statistician

- would have looked up poisson race

R – these lectures too short

- much serious statistics
- interesting fitting in the Übung (general non-linear, arbitrary function)
- most R users would
  - work in rcmdr, rstudio, ..
  - used a higher level graphics library (ggplot2, lattice)

Fitting

You are given

```
x y
0.09991595 0.031080097
0.09982738 -0.012845276
0.09946064 0.026970036
[ .. 500 .. lines]
```

and asked to make sense of it.

Start with a plot

Hint that it is of form

$y = e^{-\beta x} \sin(2\pi\omega x + \varphi)$

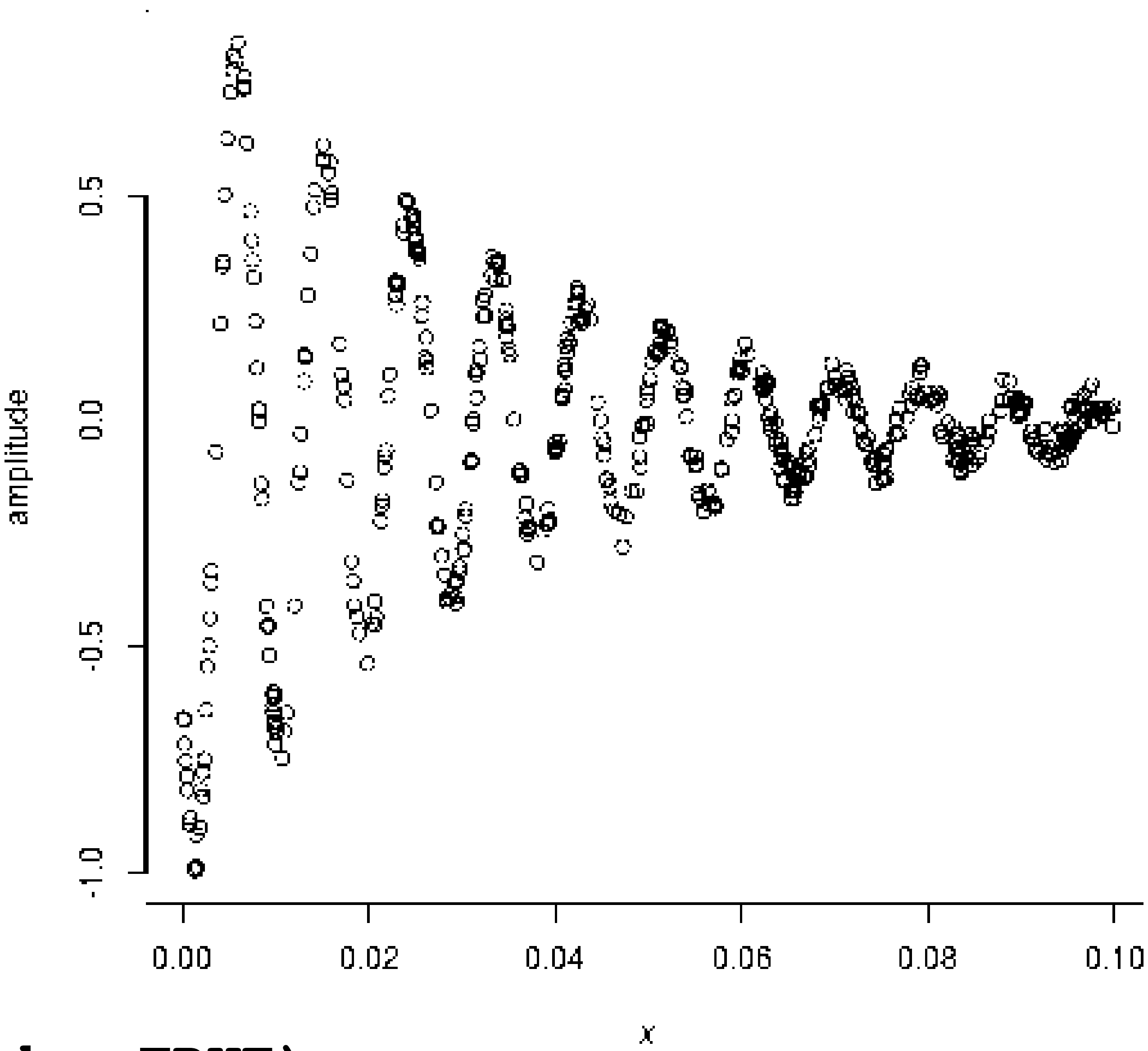
for some

β decay rate

ω frequency

φ phase

- there is noise
- points are not evenly spaced




```
d <- read.table (datafile, header=TRUE)
plot (d$x, d$y, xlab = "x", ylab="amplitude")
```

Code up the likely function

```
sinexp <- function (x, phase, freq, decay) {
     v <- sin(2 * pi * x * freq + phase)
     v <- v * exp(-decay * x)
}
```

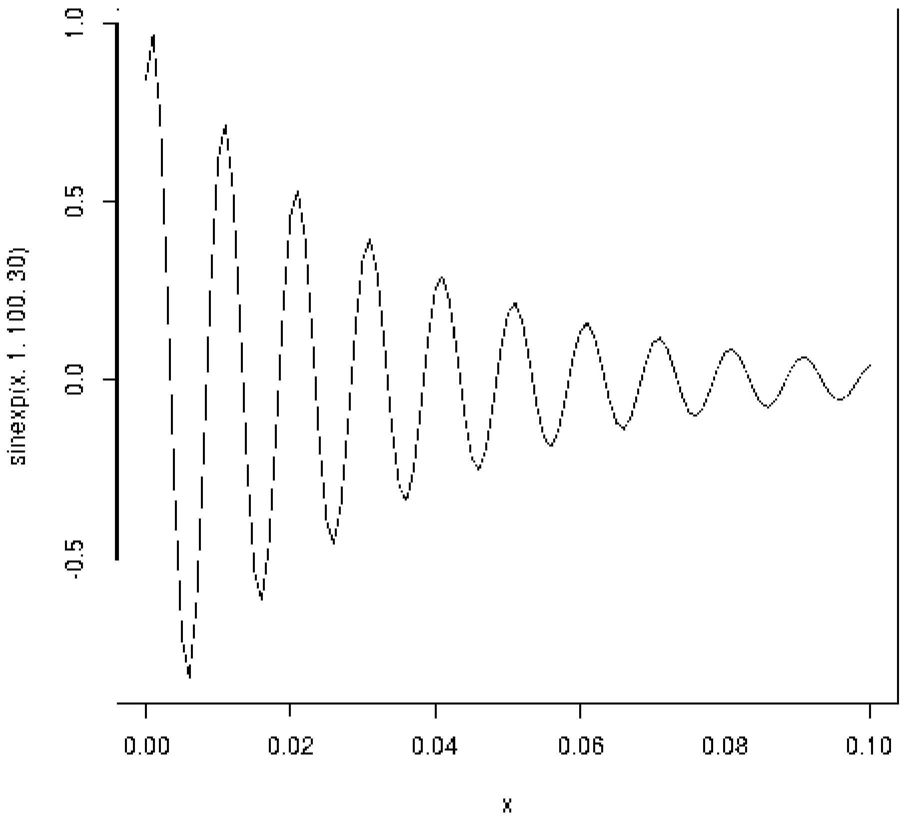
- this function acts on a vector (**x**)
- returns a vector (**v**)

Have we got the right form ?

Check if our function looks sensible

```
curve (sinexp(x, 1, 100, 30), from = 0, to = 0.1)
```

- `sinexp()` seems to be possible



Note

- `curve()` has a default name of x
- it takes guessing / experience to get sensible values
- these values can also be used as starting points for fitting

non-linear least-squares fitting

- **`?nls`**
- gives you about 330 lines of help **`?nls.control`** gives more
- what works here ? defaults including Gauss-Newton method

We are asking R to move around in 3-parameter space, β, ω, φ

```
nlmod <- nls (y~sinexp(x, phase, freq, decay), data = d,
              start = list(phase=3, freq=150, decay=30))
```

You may not have seen the **~** in R – used to define models

reading results

Results are stored in nlmod

```
> nlmod
Nonlinear regression model
  model: y ~ sinexp(x, phase, freq, decay)
   data: d
  phase    freq   decay
  3.801 108.846  30.922
 residual sum-of-squares: 0.342

Number of iterations to convergence: 10
Achieved convergence tolerance: 2.912e-06
```

accessing elements is a bore, but you can say **`coef(nlmod)[phase]`**

```
> coef(nlmod)
phase  freq decay
3.8 108.8  30.9
```

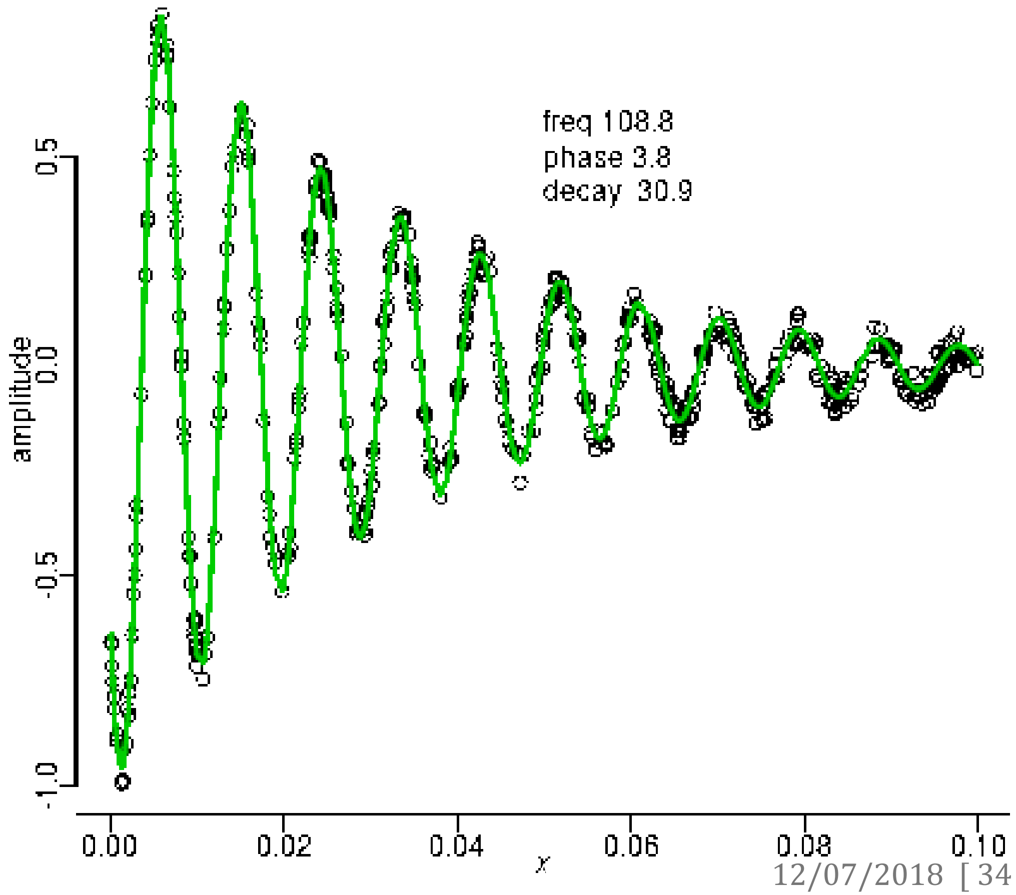
note coef() is a function call

see why coef(nlmod)[phase] works ?

to see if you really reproduce data

use the predict() function

```
plot (d$x, d$y, xlab = expression (italic(x)), ylab="amplitude")
pred=predict (nlmod)
lines(d$x, predict(nlmod), col = 3, lwd = 3)
```



Summary

- R syntax is as ugly as last week
- numerical functions remarkably
  - concise
  - simple